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Stability Relationships Between Gyrostats with Free, Constant-Speed, and Speed-Controlled Rotors

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The stability properties of a class of gyrostat systems with rotor spin rates regulated by feedback control are discussed. The concepts of proportional and rate feedback are generalized to the multiple rotor case, with the spin rate feedback gain matrix assumed to be positive definite, and that of position feedback positive semidefinite. In the past, gyrostat stability analyses have emphasized the idealized limiting case of inifinite feedback gains corresponding to a constant-speed rotor, or, in a few cases, studies of the idealized frictionless free-rotor case. The aim of this work is to study the relationship between stability results using these idealized models and the stability results obtained using the more realistic feedback controlled rotor model for the full range of feedback gains. The state space is divided into two subspaces, one of which is free of damping due to feedback (perhaps zero-dimensional), and this subspace exhibits certain appealing mathematical properties. It is shown that for the controlled systems to be asymptotically stable, a necessary and sufficient condition is that the corresponding restrained system be in the Lagrange region (stable according to Liapunov's direct method) and that the damping-free subspace be zero-dimensional (pervasive damping). Two sufficient conditions for instability are also proved. As a special case, the stability of gravity gradient gyrostat satellites with one symmetric rotor spinning around an axis fixed in the main rigid body are analyzed in detail, and numerical examples are given.

Introduction

To date most of the studies of gyrostat dynamics have made the idealized assumption that the spinning symmetric rotor(s) is perfectly maintained at a constant speed. For example, this is true of the works on gravity gradient stabilization of gyrostat satellites by Roberson and Hooker,[1] Rumiansev,[2] Stepanov,[3] Anchev,[4] and Longman et al.[5-12] This type of idealized model can be realistic if the rotor(s) is driven by a sufficiently powerful motor whose controller has a sufficiently fast time constant. At the other extreme is a model in which the rotor(s) bearings are assumed frictionless and no motor torques are applied. Such systems have been studied by Pascal,[13] Hagedorn,[14] and Otterbein.[15]

Each model is conservative, at least in the sense that the Hamiltonian is a constant of the motion. Owing to the gyroscopic nature of the dynamics, the parameter space can be divided into three regions: an unstable region, a Lagrange region for which the nonlinear equations are found stable using the Hamiltonian as a Liapunov function, and a Beletskii-Delp region often described as stable owing to gyroscopic coupling. References 13 and 14 establish certain relationships between these regions for these two types of idealized models—the free and the restrained models.

A much more realistic approach is to consider that the rotor(s) is regulated by a feedback control system. Recently, Magnus[16] discussed this problem, but as pointed out by W. Teschner using a very simple counterexample, the development is flawed. Hagedorn[17] discusses the controllability of a special kind of controlled rotor system with asynchronous drive. His work is not directly applicable to the gravity gradient gyrostat satellite problem.

Here a general controlled gyrostat system is treated which is applicable to the gravity gradient gyrostat problem. The control system consists of a positive definite spin-rate feedback matrix for the set of controlled rotors, and a positive semidefinite (possibly null) rotor angular position feedback. Since the rate feedback introduces damping, the stability properties of this realistic model are quite different from the above conservative free and restrained models. The nature of the damping generated is studied using an observability matrix,[18] the kernel of which identifies a damping-free subspace of the state space. The matrix represents a simple and precise means of determining pervasiveness of the damping.

The major aims of this paper are to assess the stability properties of the controlled rotor gyrostats, and to find relationships between these properties for controlled systems and the conclusions of Liapunov stability, instability, or Delp-region-type stability from the simpler free or restrained models. Necessary and sufficient conditions for asymptotic stability are given, and two sets of sufficient conditions for instability are given. It is interesting to note that these results are independent of the specific values of the control feedback gains. The simpler special case of complete natural damping in the essential coordinates is discussed; certain stronger stability results are given for the case of a single rotor in a rigid gravity gradient gyrostat satellite; and a new relationship between stability of free and restrained systems is given.

Gyrostats with Speed-Controlled Rotors

Consider a matrix second-order system

$$M\ddot{y}+(G+D)\dot{y}+Ky+F(\dot{y},y)=0 \tag{1}$$

with $M=M^T>0$, $G=-G^T$, $D=D^T\geq 0$, $K=K^T$, and $F(\dot{y},y)$ representing higher-order terms in $\dot{y}$ and y.

Assume the column matrix y, $F(\dot{y},y)$, and the coefficient matrices can be partitioned as

$$y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad M=\begin{bmatrix} I & m_2 & m_3 \\ m_2^T & \mu_{22} & \mu_{23} \\ m_3^T & \mu_{23}^T & \mu_{33} \end{bmatrix}$$

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$$G=\begin{bmatrix} \mathcal{G} & g_2 & g_3 \\ -g_2^T & 0 & 0 \\ -g_3^T & 0 & 0 \end{bmatrix} \quad D=\begin{bmatrix} \mathfrak{D} & 0 & 0 \\ 0 & \delta_{22} & \delta_{23} \\ 0 & \delta_{23}^T & \delta_{33} \end{bmatrix} \tag{2}$$

$$K=\begin{bmatrix} \mathcal{K} & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad F(\dot{y},y)=\begin{bmatrix} F_1(\dot{y},y_1) \\ F_2(\dot{y},y_1,y_2) \\ F_3(\dot{y},y_1) \end{bmatrix}$$

Physically, this can correspond to a gyrostat with any specified number of speed-controlled rotors. Define δ as the matrix with partitions δ_{22}, δ_{23}, δ_{33} given above. Then matrices δ and k contain the spin-rate and position feedback gains for the rotor controller. The partition y_2 contains the deviation from the nominal value for rotor spin angle for each rotor which has both proportional and rate feedback control, and y_3 is the same for those rotors which receive rate feedback control only. The remaining coordinates form y_1. In Ref. 14, y_1 is called the matrix of essential coordinates, while y_2 and y_3 are the ignorable coordinates. Due to symmetry of the rotors, the spin angular position variables have no influence on the system dynamics except through the position feedback ky_2 and their possible nonlinear effects included in $F_2(\dot{y}, y_1, y_2)$. The spin rates $\dot{y}_2$ and $\dot{y}_3$ are present only in the feedback, in the gyroscopic forces, and in the higher-order terms.

It will be assumed that $\delta>0$ and $k>0$, which implies that the rotor control alone, uncoupled from the essential coordinates [i.e., with y_1 set to zero in Eq. (1)], is asymptotically stable in variables y_2, $\dot{y}_2$, and $\dot{y}_3$, using an appropriate Liapunov function. It will also be assumed that $|\mathcal{K}|\neq 0$.

Equation (1) can be expressed in state variable form

$$\dot{x}=Ax+f(x)$$

by defining

$$x=\begin{bmatrix} \dot{y} \\ y_1 \\ y_2 \end{bmatrix} \quad f(x)=\begin{bmatrix} -M^{-1}F(\dot{y},y) \\ \hline 0 \end{bmatrix}$$

$$A=\tfrac{1}{2}R^{-1}\left[\begin{array}{c|cc} -(G+D) & \begin{matrix} -\mathcal{K} & 0 \\ 0 & -k \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} \mathcal{K} & 0 & 0 \\ 0 & k & 0 \end{matrix} & 0 \end{array}\right] \tag{3}$$

$$R=\tfrac{1}{2}\left[\begin{array}{c|cc} M & 0 & \\ \hline 0 & \mathcal{K} & 0 \\ & 0 & k \end{array}\right] \quad S=\tfrac{1}{2}\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where the fact that y_3 appears in Eq. (1) only in differentiated form has been used.

Stability of this nonlinear system will be inferred from the asymptotic stability or exponential instability of its linearized form

$$\dot{x}=Ax \tag{4}$$

which is equivalent to Eq. (1) with $F=0$.

Note that R and S are symmetric, $S\geq 0$, $|R|\neq 0$, $|A|\neq 0$, and that R and S satisfy the Liapunov equation

$$A^TR+RA=-S \tag{5}$$

which can be verified by direct substitution. The quadratic form x^TRx is thus a candidate Liapunov function representing the Hamiltonian for Eq. (4), and x^TSx represents a Rayleigh dissipation function.

The system damping is provided by two sources: the natural damping in the essential coordinates represented by the $\mathfrak{D}$ partition of D in S, and the damping due to rate feedback in the controller represented by δ. For the moment, neglect the former so that the system damping is not complete, i.e., D is not positive definite. System stability is then dependent on whether all nontrivial solutions of Eq. (4) are subject to damping, i.e., whether the system is pervasively damped. The existence of regions of the state space for which no damping occurs along system trajectories will be studied as a function of the feedback gain matrices δ and k by use of the observability matrix

$$Q^T=[S\,|A^TS|\cdots|A^{T^{n-1}}S] \tag{6}$$

when $n=\dim x$. It will be seen that the feedback gains cannot be adjusted to eliminate such a region when it exists, since a system which is not pervasively damped for one set of gains will fail to be pervasively damped for all feedback gains.

Lemma 1. Let $x(t)$ be a nontrivial solution of Eq. (4); then the following three assertions are equivalent: (i) There exists a t_0 such that $Qx(t_0)=0$; (ii) $Qx(t)\equiv 0$; (iii) $x^T(t)Sx(t)\equiv 0$.

Proof: If (iii) holds, then $Sx(t)\equiv 0$ since $S\geq 0$. Differentiating $n-1$ times and using Eq. (4) establishes that $SA^{j-1}x(t)\equiv 0$, $j=1,2,\ldots,n$, which are the partitions of $Qx(t)$. Hence (ii) holds, and (i) follows. If (i) holds, then $Qx(t_0)=0$ implies by Eq. (6) that $SA^{j-1}x(t_0)=0$, $j=1,2,\ldots,n$, and this remains valid for all $j>0$ by the Cayley-Hamilton theorem. Hence $Sx(t)=Se^{At}x(t_0)\equiv 0$ and (iii) is established.

This Lemma is generally applicable to any linear system (4) with arbitrary A matrix and associated Liapunov equation (5), and is not restricted to the specific form given in Eq. (3). Two important properties follow from this Lemma.

Property 1. The kernel or null space of Q is an A-invariant subspace contained in ker S. In other words, if at t_0 a solution to Eq. (4), $x(t_0)$, is contained in ker Q, then it is in ker Q and hence ker S for all time, and the solution $x(t)$ may be said to be in ker Q.

Property 2. Any $x(t)\in$ ker Q is free of damping. Conversely, $x(t)\notin$ ker Q implies that $x^T(t)Sx(t)\equiv 0$, and this holds on all time intervals no matter how small. Hence such an $x(t)$ is always subject to damping. Therefore ker Q represents precisely the damping free subspace. The system is pervasively damped if and only if Q is of full rank.

Now let us specialize the system to the gyrostat represented by Eqs. (3) and (4), and consider an $x(t)\in$ ker Q. Using $Sx(t)\equiv 0$ and the form of S with $\delta>0$ gives $\dot{y}_2(t)\equiv\dot{y}_3(t)\equiv 0$. Let $y_2(t)\equiv y_{20}$ and Eq. (4) becomes

$$I\ddot{y}_1+\mathcal{G}\dot{y}_1+\mathcal{K}y_1=0$$

$$m_2^T\ddot{y}_1-g_2^T\dot{y}_1+ky_{20}=0$$

$$m_3^T\ddot{y}_1-g_3^T\dot{y}_1=0$$

Since $|\mathcal{K}|\neq 0$, the first equation has no eigenvalue equal to zero, hence it has no solutions that are polynomials in t. Then these equations require $ky_{20}=0$, which implies $y_{20}=0$ since $k>0$. Therefore we have the following:

Property 3. Any solution $x(t)\in$ ker Q of the controlled gyrostat equations (3) and (4) has the property that all rotors spin at the nominal constant speed, and the control system applies no torques. This $x(t)$ is also a solution of the equations for the corresponding free or restrained systems as defined in the following section.

Consider a new system of the form equations (3) and (4), differing only in that new feedback gains $\delta'>0$ and $k'>0$ are used, and let $S'\geq 0$ and Q' be the corresponding new S and Q

matrices. Since any $x(t) \notin \ker Q$ has $y_2(t) \equiv \dot{y}_3(t) \equiv 0$, this $x(t)$ will satisfy the new system equations as well. Clearly, $S'x(t) \equiv 0$, and this implies $Q'x(t) \equiv 0$. In addition, this establishes $\ker Q \subset \ker Q'$ and by symmetry $\ker Q' \subset \ker Q$.

Property 4. Any solution $x(t) \in \ker Q$ of the controlled gyrostat equations (3) and (4) is also a solution, and one free of damping, for all choices of feedback gains $\delta > 0$ and $k > 0$. Furthermore, the damping-free subspace $\ker Q$ is invariant under changes in $\delta > 0$ and $k > 0$, and represents a common characteristic subspace for the restrained system, the free system, and all controlled systems with positive definite feedback gains.

Stability of Free and Restrained Gyrostats

Consider the two limiting cases of Eq. (1) with $F=0$ obtained by setting the feedback gains δ and k to zero or to infinity so that the rotors can be considered as freely rotating and frictionless or constrained to a constant speed. In the latter case, let $y_2(t) \equiv y_3(t) \equiv 0$ in Eqs. (3) and (4) to obtain the restrained system

$$I\ddot{y}_1 + \mathcal{G}\dot{y}_1 + \mathcal{K}y_1 = 0 \tag{7}$$

Note that if $x(t) \in \ker Q$ is a solution of Eq. (4), then the associated y_2 and $\dot{y}_3$ are zero and y_1 satisfies Eq. (7) as well, by Property 3. Since by Property 4 the $\ker Q$ is independent of the feedback gains, the solution $y_1(t)$ of the restrained system (7) can be said to be in $\ker Q$.

In the free gyrostat case the system has extra degrees of freedom associated with the motion of the unrestrained rotors, and hence has additional eigenvalues which are equal to zero. In order to produce a free system as a limiting case of Eq. (4), we should require that when $y_1 = \dot{y}_1 = 0$, $\dot{y}_2$ and $\dot{y}_3$ are zero; i.e., when the essential coordinates are in equilibrium, the rotors are spinning at the nominal rates. These constraints together with the first integrals associated with the cyclic coordinates give the free system associated with Eq. (4) as

$$I_F\ddot{y}_1 + \mathcal{G}_F\dot{y}_1 + \mathcal{K}_F y_1 = 0 \tag{8}$$

with

$$I_F = I - m\mu^{-1}m^T$$

$$\mathcal{G}_F = \mathcal{G} + m\mu^{-1}g^T - g\mu^{-1}m^T$$

$$\mathcal{K}_F = \mathcal{K} + g\mu^{-1}g^T$$

where μ is the partition of m containing μ_{22}, μ_{23}, μ_{33}, and $m = [m_2\ m_3]$, $g = [g_2\ g_3]$. For practical problems $I_F > 0$, which is assumed here. Note that if $x(t) \in \ker Q$ is a solution of Eq. (4), then by Property 3, $y_2(t) \equiv \dot{y}_3(t) \equiv 0$ and the limiting case conditions above are satisfied, and $y_1(t)$ from Eq. (4) satisfies Eq. (8). Such a $y_1(t)$ of the free system will be said to be in $\ker Q$.

Stability of free and restrained systems are discussed in Refs. 13-15 and the main results applied to the linear equations at hand are:

Result 1. The following four statements hold:

(i) The free and restrained systems (7) and (8) cannot be asymptotically stable, since there is no damping present.

(ii) If $\mathcal{K} > 0$ then Eq. (7) is Liapunov stable in y_1, $\dot{y}_1$ using Hamiltonian $\dot{y}_1^T I\dot{y}_1 + y_1^T\mathcal{K}y_1$ as a Liapunov function. The analogous statement applied to Eq. (8) when $\mathcal{K}_F > 0$. When these conditions hold, the system (7) [or (8)] is said to be in the Lagrange stability region of the parameter space. In the event that $\mathcal{K} \not> 0$ (or $\mathcal{K}_F \not> 0$), but the linear equation (7) [or (8)] is stable, it is said to be in the Beletskii-Delp stability region associated with stability due to gyroscopic coupling.

(iii) $\mathcal{K} > 0$ implies $\mathcal{K}_F > 0$ since $g\mu^{-1}g^T$ is positive semidefinite; i.e., if the restrained system (7) is in the Lagrange region, so is the free system.

(iv) If $\mathcal{K}$ [or $\mathcal{K}_F$] has an odd number of negative eigenvalues, then Eq. (7) [or (8)] is unstable, since the constant term $\det(\mathcal{K})$ of the characteristic equation $\det(I\lambda^2 + \mathcal{G}\lambda + \mathcal{K}) = 0$ is the product of the eigenvalues and hence negative.

Result 2. From Ref. 14, if that part of the Hamiltonian free of generalized momenta (H_0) has a strict relative maximum when the generalized coordinates $y_1 = 0$, the system is unstable. The Lagrangian $L = \frac{1}{2}(\dot{y}_1^T I\dot{y}_1 - y_1^T\mathcal{K}y_1 + 2\dot{y}_1\mathcal{G}y_1)$ produces $H_0 = \frac{1}{2}y_1^T(\mathcal{K} + \mathcal{G}^T I^{-1}\mathcal{G})$. Hence, if $\mathcal{K} + \mathcal{G}^T I^{-1}\mathcal{G} < 0$, then system (7) is unstable, and similarly for system (8).

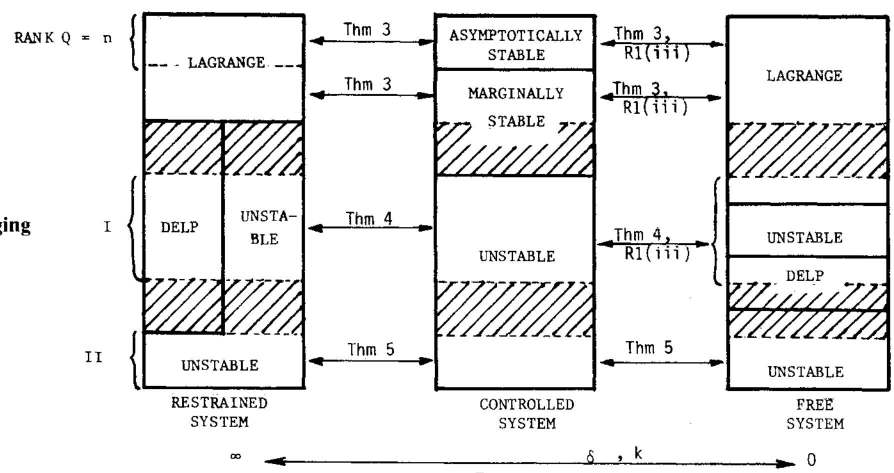
A third new result is as follows:

Result 3. If $\dim y_2 + \dim y_3 < \bar{n}$, where $\bar{n}$ is the number of negative eigenvalues of $\mathcal{K}$, then $\mathcal{K}_F$ cannot be positive definite and the free system (8) is not in the Lagrange region.

Proof: Consider the following two algebraic equations in a matrix variable z whose dimension is $\dim y_1$:

$$\sqrt{\mu^{-1}}g^T z = 0 \qquad Cz = 0$$

where the rows of C are linearly independent eigenvectors corresponding to all nonnegative eigenvalues. The first equation represents $\dim y_2 + \dim y_3$ scalar equations, while the second has $\dim y_1 - \bar{n}$ scalar equations. When $\dim y_2 + \dim y_3 < \bar{n}$, the number of equations is less than the



I. The Region where $\dot{y}_1^T I \dot{y}_1 + y_1^T K y_1 \geq 0$ for All $x \in \ker Q$

II. The Region Where There Exist Unbounded Solutions in ker Q

Fig. 1 Stability evolutions with changing feedback gains.

number of unknowns in z, so that a nontrivial solution for z exists. Using such a z, note that

$$z^T \mathcal{K}_F z = z^T \mathcal{K} z + (\sqrt{\mu^{-1}} g^T z)^T (\sqrt{\mu^{-1}} g^T z) = z^T \mathcal{K} z$$

and that $z^T \mathcal{K} z < 0$ by definition of C. Hence $z^T \mathcal{K}_F z < 0$ and $\mathcal{K}_F$ is not positive definite. ∎

By way of interpretation, note that dim y_2 + dim y_3 is the number of rotors, and whether $\bar{n}$ is odd or even is related to whether the restrained system (7) is in the Delp or the unstable regions by Result 1 (iv). As a special case, a restrained gyrostat with only one rotor with parameters in the Delp region (so that $\bar{n} \geq 2$ and $\bar{n}$ is even) satisfies the conditions of Result 3, and therefore the associated free system cannot be in the Lagrange region; i.e., these two regions of the parameter space are disjoint.

Stability Relationships

In this section we shall establish relationships between the stability properties of controlled systems and their restrained and free system counterparts. First, let us generate a stability theorem and an instability theorem applicable to general linear time-invariant systems in which the "damping" is not complete. Consider a system of the form (4) with $S \geq 0$, R and Q satisfying Eqs. (5) and (6), but with A, R, and S not restricted to the form given in Eq. (3).

Theorem 1. If $R > 0$, then system (4) is Liapunov stable and for Eq. (4) to be asymptotically stable [and any possibly nonlinear system (1) associated with Eq. (4) as well] a necessary and sufficient condition is that

$$\text{rank} \quad Q = n$$

where $n = \dim x$.

The first part of the theorem is a standard Liapunov result, and for the second part see Ref. 18.

Theorem 2. (i) If there exists an x_0 such that $x_0^T R x_0 < 0$, then system (4) cannot be asymptoticallys table; (ii) furthermore, if $x^T R x \geq 0$ for all $x \in$ ker Q, then Eq. (4) is unstable.‡

Proof: (i) Let $x(t)$ be the solution of Eq. (4) and $x(t_0) = x_0$. Then

$$x^T(t) R x(t) = x_0^T R x_0 - \int_0^t x^T(\tau) S x(\tau) \mathrm{d}\tau$$

$$\leq x_0^T R x_0 < 0 \qquad \forall t \geq t_0$$

by Eq. (5), $S \geq 0$ and the assumption. Hence $x(t)$ cannot approach the origin as $t \to \infty$. Therefore Eq. (4) is not asymptotically stable. (ii) Since $x_0^T R x_0 < 0$, by assumption $Q x_0 \neq 0$. Using Lemma 1 and Eq. (5), $x^T(t) S x(t) \neq 0$ and $x^T(t) R x(t)$ is monotonically decreasing.

Suppose $x(t)$ is bounded as $t \to \infty$, then so is $x^T(t) R x(t)$. There must be a constant r_0 such that $x^T(t) R x(t) \to r_0$ as $t \to \infty$. Then $x^T(t) S x(t) = \mathrm{d}/\mathrm{d}t [x^T(t) R x(t)] \to 0$ by analyticity. Hence $Sx(t) \to 0$, and by differentiation $S\dot{x}(t) = SAx(t) \to 0, \ldots, SA^{n-1} x(t) \to 0$, and therefore $Qx(t) \to 0$ or $x(t) \to$ ker Q as $t \to \infty$. For a sufficiently large t, there exists an $x^* \in$ ker Q such that $\| x(t) - x^* \|$ is arbitrarily small, and therefore $| x^T(t) R x(t) - x^{*T} R x^* |$ is arbitrarily small. But from part (i) of the proof, $x^T(t) R x(t) \leq x_0^T R x_0 < 0$, so $x^{*T} R x^* < 0$. This contradicts the assumption. So $x(t)$ is not bounded, and system (4) is unstable. ∎

Now let us return to a more specialized form of (4) representing a gyrostat system with A specified by Eq. (3), or equivalently an equation of the form (1) with $F(\dot{y}, y) = 0$.

Theorem 3. If the restrained system (7) is in the Lagrange region, then the controlled system (4) is Liapunov stable, and this result applies independent of the actual feedback gains $\delta > 0$ and $k > 0$. For asymptotic stability of a Liapunov stable system (4), a necessary and sufficient condition is

$$\text{rank } Q = n$$

where $n = \dim x = 2$ (dim y_1 + dim y_2) + dim y_3. Q can be calculated from Eq. (6) for any arbitrarily chosen feedback gains $\delta > 0$ and $k > 0$.

Proof: If Eq. (7) is in the Lagrange region, then by definition $\mathcal{K} > 0$. Since $M > 0$ and $k > 0$, we have $R > 0$ in Eq. (3). Hence Theorem 1 can be applied directly. According to Property 5, ker Q is invariant with respect to δ and k, so is rank Q; therefore this rank condition is unaltered by changes of the feedback gains. ∎

Theorem 4. (i) If the restrained system (7) or the free system (8) is unstable or is in the Delp region, then the controlled system (4) for all feedback gains $\delta > 0$ and $k > 0$ fails to be asymptotically stable. (ii) Furthermore, if $\dot{y}_1^T I \dot{y}_1 + y_1^T \mathcal{K} y_1 \geq 0$ for all $x \in$ ker Q, then all controlled systems are unstable independent of the gains $\delta > 0$, $k > 0$.

Proof: (i) If Eq. (7) or Eq. (8) is unstable or in the Delp region, according to the definition of the Lagange region and acccording to Result 1, $\mathcal{K}$ is not positive definite. By assumption $|\mathcal{K}| \neq 0$, so there exists a y_1 such that $y_1^T \mathcal{K} y_1 < 0$. Hence take $x^T = [0 \mid y_1^T \; 0]$ and then $x^T R x < 0$. By Theorem 2 (i), the controlled system is not asymptotically stable, and this holds for all positive definite gains. (ii) By Properties 1 and 3, if $x \in$ ker Q, then $\dot{y}_2 = \dot{y}_3 = y_2 = 0$, and therefore $x^T R x = \dot{y}_1^T I \dot{y}_1 + y_1^T \mathcal{K} y_1 \geq 0$. So Theorem 2(ii) can be applied. ∎

Theorem 5. If there is in ker Q an unbounded solution of the restrained system (7) or the free system (8), or the controlled system (4) for any set of positive definite gains, then all these systems are unstable.

Proof: By Property 3, any solution $x(t) \in$ ker Q of Eq. (4) for any δ and k is a solution of Eqs. (7) and (8) and (4) for all positive definite gains. On the other hand, if $y_1(t) \in$ ker Q is a solution of Eqs. (7) or (8), by its definition, $x(t) = [\dot{y}_1^T(t) \; 0 \; 0 \mid y_1^T(t) \; 0]^T \in$ ker Q is a solution of Eq. (4) for all positive definite gains. Therefore any unbounded solution in ker Q of Eqs. (7) or (8) or (4) for all positive definite gains is an unbounded solution for all other systems and all three systems are unstable. ∎

Figure 1 summarizes the results obtained, indicating how the stability properties evolve in the parameter space with changing feedback gains δ and k going from the restrained to the free system.

The two boxes on the left and right represent the parameter spaces for the restrained and free systems, respectively. Each of them is divided by solid lines into Lagrange, Delp, and unstable regions. These regions are further divided by the dashed lines according to the characteristics I and II stated in

‡The condition for instability presented in Ref. 18 appears to be insufficient. For a counterexample, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} -1 & -1 \\ -1 & -0.5 \end{bmatrix} \mathcal{K} = [0 \;\; 1 \;\; 0 \;\; 0]^T$$

then

$$Q^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} x_0 = Q^T \mathcal{X} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and $x_0^T R x_0 = -0.5 < 0$. But the system is Liapunov stable.

the figure. The box in the middle of Fig. 1 represents the same parameter space (exclusive of δ and k, of course) for the controlled system, and is divided by solid lines into asymptotic stability, marginal stability, and instability regions. Points from each region are mapped horizontally from one system to another, according to the theorem or rule stated, as the feedback gains are varied from ∞ to 0.

The Lagrange region of the restrained system consists of two parts: the one in which rank $Q=n$ maps into the asymptotic stability region of the controlled system, and the second part maps into part of the marginal stability region according to Theorem 3. Both of them finally form part of the Lagrange region of the free system by Result 1 (iii) when the feedback gains go to zero.

For the Delp and unstable region of the restrained system, the associated controlled system cannot be asymptotically stable by Theorem 4 (i). The two regions where the sufficient conditions for instability of the controlled system in Theorems 4 (ii) and 5 are satisfied are indicated by I and II, respectively. The same has been done for the free system, by use of Result 3, and we may equivalently characterize region I by $y_I^T I_F y_I + y_I^T \mathcal{K}_F y_I \geq 0$ for all $x \in \ker Q$. But this time instability of the controlled system can become stability in the Lagrange region of the free system as the feedback gains go to zero.

As mentioned before, all of the boundaries of the various stability areas for the controlled systems associated with necessary and sufficient or only sufficient conditions are independent of the actual feedback gains.

No information was obtained to indicate how the shaded areas for the restrained, controlled, and free systems map from one to another, although in certain special cases discussed in the next section results will be obtained.

Finally, consider the effect of any natural damping in the essential coordinates. Assume $\mathfrak{D}>0$, then $D>0$, i.e., the system damping is complete. It is easy to check that in this case rank $Q=n$ and $\ker Q=0$. So Theorems 1 and 2 reduce to the following much simpler results.

Corollary 1. Suppose there is complete damping $\mathfrak{D}>0$ in the essential coordinates. If the restrained system is in the Lagrange region, then the controlled system for all positive definite gains is asymptotically stable.

Corollary 2. Suppose $\mathfrak{D}>0$. If the restrained system or the free system is in the Delp region or in the unstable region, then the controlled system for all positive definite gains is unstable.

Gravity Gradient Gyrostat Satellites

To illustrate the previous theory, consider a gyrostat satellite composed of a rigid body and one symmetric rotor whose rotation rate and angle are controlled by a feedback system. Assume the satellite to be in a circular orbit and subject to gravity gradient torques, and, for simplicity, take the orbital angular velocity as unity. The corresponding constant speed rotor problem has been treated in Refs. 1-12.

Using the same methods and notation as in Ref. 12, one can derive the linearized matrix second-order differntial equations of motion for the satellite, which has the form of Eq. (1) with $F(\dot{y},y)$ and

$$y=\begin{bmatrix}\theta\\ \theta_4\end{bmatrix}\quad M=\begin{bmatrix}I & \mu\alpha\\ \mu\alpha^T & \mu\end{bmatrix}\quad G=\begin{bmatrix}\mathcal{G} & \mu\tilde{\xi}_2\alpha\\ -\mu(\tilde{\xi}_2\alpha)^T & 0\end{bmatrix} \tag{9}$$

$$D=\begin{bmatrix}\mathfrak{D} & 0\\ 0 & \delta\end{bmatrix}\quad K=\begin{bmatrix}\mathcal{K} & 0\\ 0 & k\end{bmatrix}$$

where $\theta=[\theta_1\theta_2\theta_3]^T$ are Euler angles representing small deviations of the body axes relative to the equilibrium orientation; θ_4 is the difference between the spin angle of the rotor relative to the body and the desired spin angle which has a constant rate $\dot{\theta}_d$; I is the inertia matrix of the satellite (including the rotor) in the body axis system; μ is the inertia of the rotor about its spin axis; and α is a unit vector along the spin axis written in matrix form with components relative to body axis coordinates. The matrices ξ_2, ξ_3 consist of the components of the unit vector along the orbital angular velocity and along the local upward vertical when the body is in the equilibrium orientation, and $\tilde{\xi}_2$ is a matrix such that $\tilde{\xi}_2\alpha$ for any vector α gives the matrix representing the cross-product between ξ_2 and α. The scalars $\delta>0$ and $k\geq 0$ are spin rate and spin angle feedback gains. Matrices $\mathcal{G}$ and $\mathcal{K}$ are skew-symmetric and symmetric, respectively, and are given in terms of the parameters by (Ref. 12),

$$\mathcal{G}=I\tilde{\xi}_2+\tilde{\xi}_2 I-\widetilde{I\xi_2}-\dot{\theta}_d\tilde{\alpha}$$

$$\mathcal{K}=(\tilde{\xi}_2 I-\widetilde{I\xi_2}-\dot{\theta}_1\tilde{\alpha})\xi_2-3(\tilde{\xi}_3 I-\widetilde{I\xi_3})\xi_3$$

Note that the coefficient matrix $I_F=I-\mu\alpha\alpha^T$ in the free system (8) is the inertia matrix of the gyrostat with the spin inertial moment of the rotor removed. Therefore $I_F>0$ by its physical meaning, and $M>0$.

Assume $|\mathcal{K}|\neq 0$ and $\mathfrak{D}=0$, then all previous results can be applied here. For this comparatively simple case, additional insight as well as general properties can be obtained by examining the characteristic equation and its root locus. This equation for Eq. (1) with Eq. (9) and $F=0$ is

$$\begin{vmatrix} I\lambda^2+\mathcal{G}\lambda+\mathcal{K} & \mu\lambda^2\alpha+\mu\lambda\tilde{\xi}_2\alpha\\ \mu\lambda^2\alpha^T-\mu\lambda(\tilde{\xi}_2\alpha)^T & \mu\lambda^2+\delta\lambda+k\end{vmatrix}=0$$

which can be split into two determinants, one of which has a second row $[0\ \ \delta\lambda+k]$, and the second can be reduced by Theorem 1.2 in Ref. 19 to yield the characteristic polynomial

$$p(\lambda;\ \delta,k)\triangleq\mu\lambda^2 P_F(\lambda)+(\delta\lambda+k)P_R(\lambda)=0 \tag{10}$$

with

$$P_F(\lambda)=|I_F\lambda^2+\mathcal{G}_F\lambda+\mathcal{K}_F|$$

$$P_R(\lambda)=|I\lambda^2+\mathcal{G}\lambda+\mathcal{K}|$$

where P_F and P_R happen to be the characteristic polynomials of the free and the restrained systems, respectively. Using this special form one can obtain additional stability properties applicable to gyrostat satellites with one rotor.

Property 6. If for some $\delta_I>0$ and $k_I\geq 0$ the controlled system has an imaginary eigenvalue $j\omega\neq 0$ with multiplicity m, so do the restrained system, the free system, and the controlled systems for all $\delta>0$ and $k\geq 0$.

To prove it, use induction and separate the real and imaginary parts of Eq. (10) and its derivatives with respect to λ.

Define a system to be marginally unstable when the instability is due to repeated eigenvalues on the imaginary axis.

Property 7. If for some $\delta_I>0$ and $k_I\geq 0$ the controlled system is asymptotically stable (marginally stable, marginally unstable, or exponentially unstable), so is the controlled system for all $\delta>0$ and $k\geq 0$.

Proof: Suppose $P(\lambda;\ \delta_I,\ k_I)$ is asymptotically stable, but $P(\lambda;\ \delta_2,\ k_2)$ is not, where $\delta_2>0$, $k_2\geq 0$. From continuity of the roots with respect to the coefficients, there must be a pair of gains $\delta_3>0$ and $k_3\geq 0$ such that $P(\lambda;\ \delta_3,\ k_3)$ has at least one root on the imaginary axis. From Eq. (10) and the assumption $|\mathcal{K}|\neq 0$, this root cannot be zero. According to Property 6, $P(\lambda;\ \delta_I,\ k_I)$ should have the same imaginary root, which is a contradiction.

Suppose $P(\lambda;\ \delta_I,\ k_I)$ is exponentially unstable, but $P(\lambda;\ \delta_3,\ k_2)$ is not. Then there exists a pair of gains δ_3 and k_3 such that $P(\lambda; \delta_3, k_3)$ has some imaginary roots, but no root in the open right half plane and for fixed δ_3 and k_3 there exists another pair of gains δ_4 and k_4 such that $|\delta_3 - \delta_4|$ and $|k_3 - k_4|$ are arbitrarily small and $P(\lambda; \delta_4, k_4)$ has at least one root in the open right half plane. By Property 6, $P(\lambda; \delta_4, k_4)$ also has the same imaginary roots with the same multiplicity as $P(\lambda; \delta_3, k_3)$. After eliminating the associated common divisors from $P(\lambda; \delta_3, k_3)$ and $P(\lambda; \delta_4, k_4)$, we get two polynomials. One of them has all roots in the open left half plane, and the other has some roots in the open right half plane, which contradicts continuity, since δ_4, k_4 can be made arbitrarily close to δ_3, k_3.

The marginal stability and marginal instability cases are easily proved by Property 6. ■

Based on this property, one can say that the controlled system is simply asymptotically stable (marginally stable, etc.) without regard for the specific values of the feedback gains δ and k.

The root locus for varying $\delta > 0$ and $k \geq 0$ as δ or k approaches infinity (as δ and k approach zero) has certain of the eigenvalues of the controlled system approaching the eigenvalues of the restrained (free) system, as is seen from Eq. (10). With these new properties, stronger stability relationships are obtained for gyrostat satellites with one rotor as follows:

Corollary 3. If the free or restrained system is exponentially unstable, so is the controlled system (from continuity and Property 7).

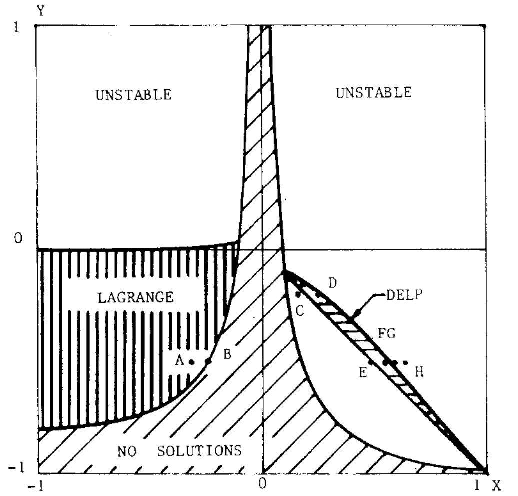


Fig. 2 Stability boundary plot for the restrained system.

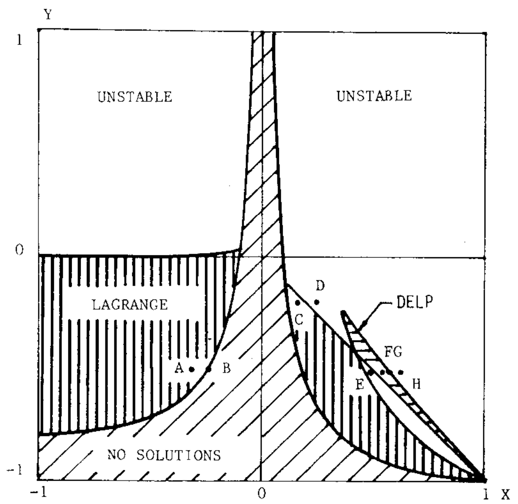


Fig. 3 Stability boundary plot for the free system.

Corollary 4. If the controlled system is marginally unstable, so are the free and the restrained systems (from Property 6 and Corollary 3).

Corollary 5. If the controlled system is marginally stable, then the free and restrained systems are stable (i.e., in the Lagrange or Delp region) or marginally unstable (from Corollary 3).

Finally, apply Result 3 to one rotor gyrostat satellites and get

Corollary 6. If the restrained system is in the Delp region, then the free system cannot be in the Lagrange region.

Let us illustrate these properties with certain numerical examples. Let

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \quad \alpha = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$
$$\xi_2 = \begin{bmatrix} \sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \cos\phi = \frac{\mu \dot{\theta}_d}{I_1 - I_2} \tag{11}$$

which corresponds to case 3a in Ref. 12, i.e., the rotor axis is along a principal axis, and the local vertical is aligned with a principal axis. Take $\mu / I_2 = 0.3$, $\mu \dot{\theta}_d / I_2 = 0.1$, and $k = 0$; and parameterize I_i as

$$\frac{I_1}{I_2} = \frac{1 - X}{1 + XY} \quad \frac{I_3}{I_2} = \frac{1 + Y}{1 + XY} \quad |X|, |Y| \leq 1$$

The stability boundary plots in parameter X-Y plane for the restrained and the free system are given in Figs. 2 and 3, respectively. From these and other plots in Refs. 12-15, we observe that any restrained system in the Delp region cannot be converted into a free system in the Lagrange region by letting the feedback gains go to zero. This may be interpreted by using Result 6. Here the number of rotors (1) is less than the number of negative eigenvalues of $\mathcal{K}$ (2) for a single rotor system in the Delp region.

For a representative set of points in the X-Y plane, including each type of stability region (see Figs. 2 and 3), the root loci have been generated according to Eq. (10), and the results are summarized in Table 1, and given in Figs. 4-6.

In Figs. 4-6 the root loci go from the roots of the free system (marked by X) to the roots of the restrained system (marked by O) with δ increasing from 0 to ∞. In certain cases the locus is too small to show, and is indicated by an overlapping O and X with an arrow which shows the direction of root motion as well as which side of the imaginary axis it is on.

When $\xi_2 = [0 \ 1 \ 0]^T$, we have ker $Q \neq 0$, which means the damping produced by the rate feedback is not pervasive. Actually, in this case, the equation of motion takes a very simple uncoupled form:

$$I_1 \ddot{\theta}_1 + I_3 \dot{\theta}_3 + [4(I_2 - I_3) + \mu \dot{\theta}_d] \theta_1 = 0$$
$$I_3 \ddot{\theta}_3 - I_3 \dot{\theta}_1 = 0$$
$$I_2 \dot{\theta}_2 + 3(I_1 - I_3) \theta_2 + \mu \ddot{\theta}_4 = 0$$
$$\mu \ddot{\theta}_4 + \delta \dot{\theta}_4 + \mu \ddot{\theta}_2 = 0$$

Clearly the damping-free subspace, ker Q, is $\{\theta_2 = \dot{\theta}_2 = \theta_4 = \dot{\theta}_4 = 0\}$. The eigenvalues belonging to ker Q are determined by the first two equations above, which are independent of the feedback δ. Their root loci have degenerated

Table 1 Root locus summary

Figure no.	Position in X-Y plane	Stability of the systems: Restrained	Free	Controlled	Associated theorem
4a	A	Lagrange	Lagrange	Asymptotically stable	3
4b	B	Lagrange	Lagrange	Stable	3
5a	C	Unstable	Lagrange	Unstable	4,5
5b	D	Delp	Unstable	Unstable	4,R3
6a	E	Unstable	Unstable	Unstable	4,5
6b	F	Delp	Delp	Unstable	4,R3
6c	G	Unstable	Delp	Unstable	4,5
6d	H	Unstable	Unstable	Unstable	4,5

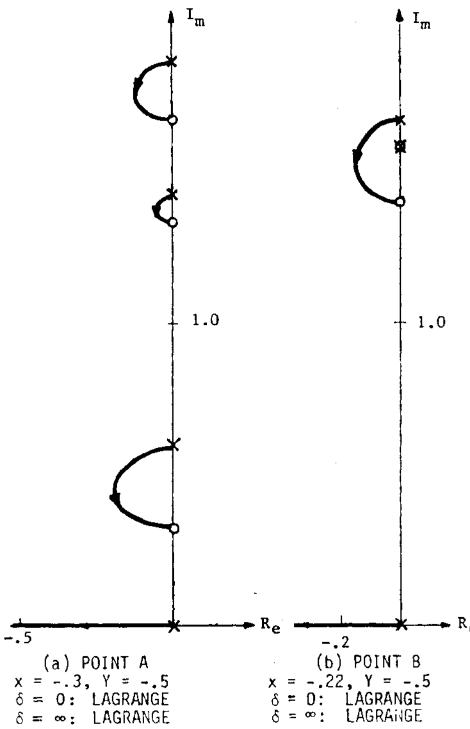


Fig. 4 Root loci for points A and B.

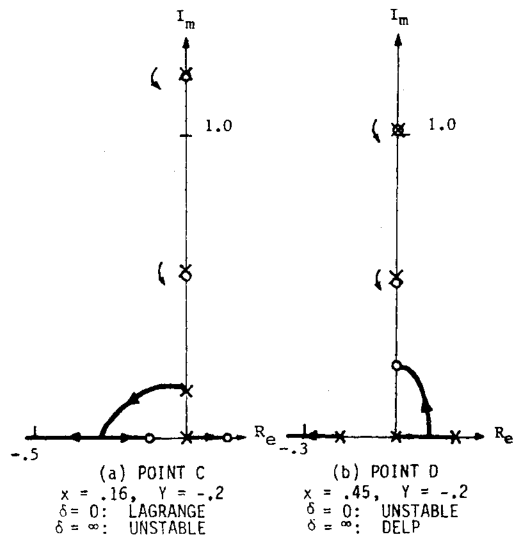


Fig. 5 Root loci for points C and D.

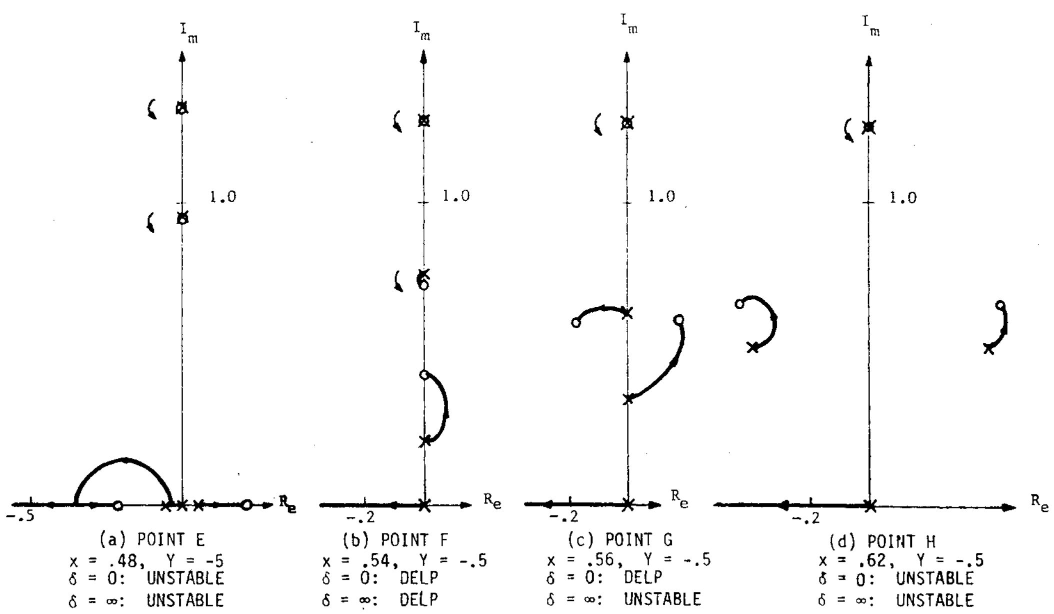


Fig. 6 Root loci for points E,F,G, and H.

to fixed points as shown in Fig. 4b, where $\xi_2 = [0\ 1\ 0]^T$. This is in accordance with Properties 3 and 4.

If there is any unbounded motion in ker Q, then some of the associated eigenvalues are in the open right half plane and their root loci are fixed points. Therefore the restrained, the free, and the controlled (for any δ) systems are unstable. This is consistent with the prediction of Theorem 5.

Suppose $\xi_2 = [0\ 1\ 0]^T$ again. If $\mathcal{K}$ is not positive definite and if $\dot\theta^T I \dot\theta + \theta^T \mathcal{K} \theta \geq 0$ holds for all $x \in$ ker Q, then k_{11} and $k_{33} > 0$ and $k_{22} = 3(I_1 - I_3) < 0$ (since $\theta_2 = \dot\theta_2 = 0$ in ker Q), where k_{11}, k_{22}, and k_{33} are elements of $\mathcal{K}$. Observe that the characteristic equation of the last two equations above is

$$\mu^2\lambda^3 - (\mu\lambda + \delta)(I_2\lambda^2 + k_{22}) = 0$$

which has unstable eigenvalues when $k_{22} < 0$. This illustrates Theorem 4 (ii).

Conclusions

In studying the damping introduced by rate feedback, a damping-free subspace has been identified and shown to be an invariant subspace with respect to feedback gains; i.e., all solutions in this subspace are common solutions of the restrained system, of the free system, and of the controlled system independent of the values of the feedback gains. The subspace is characterized as the null space or kernel of the observability matrix Q, from control theory, using the damping coefficient matrix as the observation matrix. These properties of the subspace form an easy approach to studying pervasiveness of damping in the system by determining whether ker $Q = 0$.

In relating the constant rotor speed gyrostats to the speed-controlled gyrostats, it was found that when the former is in the Lagrange region of Liapunov stability (for which the dynamic potential is positive definite, $\mathcal{K} > 0$), the controlled system is stable for all possible feedback gains, and will furthermore be asymptotically stable if and only if ker $Q = 0$. When the parameters are not in the Lagrange region (i.e., when the matrix $\mathcal{K}$ of the quadratic approximation to the dynamic potential is not positive definite), the controlled system is never asymptotically stable, so that asymptotic stability of the controlled system occurs only in that subset of the Lagrange region of the restrained problem for which the observability matrix Q has full rank. Two sufficient conditions for instability when $\mathcal{K} \not> 0$ have also been proved, which identify how certain parts of the parameter space are affected by introducing active control.

The Delp region for the restrained or the free rotor system (i.e., the region of stability of the linearized equations excluding the Lagrange region) is shown to be destabilized by nonpervasive damping if ker Q does not interest the set $\{x$: $x^T R x < 0\}$ where the dynamic potential fails to be a minimum. No general proof was found to establish that an unstable limiting system can never be stabilized by introducing speed control with nonpervasive damping, but such a proof was given in the case of an exponentially unstable rigid gravity gradient one-rotor gyrostat satellite. Of course, if the system damping is pervasive either owing to the feedback control or to natural damping of the essential coordinates, the Lagrange region of the restrained system is the asymptotic stability region for the controlled system, and all other parameters yield instability.

A new result that applies to the gravity gradient one-rotor gyrostat problem is that the Lagrange region of the free system and the Delp region of the restrained system must be disjoint.

The results obtained allow one to select parameters for a controlled gyrostat based simply on the restrained model, and to be guaranteed of appropriate stability of the controlled system. The control gains could then be optimized to maximize the degree of stability. In the example root loci, the high-frequency roots were found to have small damping for all gains.

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